

ASP-640

M.A./M.Sc. (Final) Examination, 2021

MATHEMATICS

Paper - VI

(Topology and Functional Analysis)

Time : 1½ Hours]

[Maximum Marks : 100

Section-A

(Marks : 2 × 10 = 20)

Note :- Answer all ten questions. (Answer limit 50 words). Each question carries 2 marks.

(खण्ड-अ)

(अंक : 2 × 10 = 20)

नोट :- सभी दस प्रश्नों के उत्तर दीजिए। (उत्तर-सीमा 50 शब्द)। प्रत्येक प्रश्न 2 अंक का है।

Section-B

(Marks : 4 × 5 = 20)

Note :- Answer all five questions. Each question has internal choice (Answer limit 200 words). Each question carries 4 marks.

(खण्ड-ब)

(अंक : 4 × 5 = 20)

नोट :- सभी पाँच प्रश्नों के उत्तर दीजिए। प्रत्येक प्रश्न में विकल्प का चयन कीजिए (उत्तर-सीमा 200 शब्द)। प्रत्येक प्रश्न 4 अंक का है।

Section-C

(Marks : 20 × 3 = 60)

Note :- Answer any three questions out of five (Answer limit 500 words). Each question carries 20 marks.

(खण्ड-स)

(अंक : 20 × 3 = 60)

नोट :- पाँच में से किन्हीं तीन प्रश्नों के उत्तर दीजिए (उत्तर-सीमा 500 शब्द)। प्रत्येक प्रश्न 20 अंक का है।

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(1)

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Section-A

2 each

1. Define the following :

- (i) Coarser topology
- (ii) Relative topology
- (iii) Hausdorff space
- (iv) Regular space
- (v) Open mapping
- (vi) State closed graph theorem

In a complex inner product space X , show that :

- (vii) $(x, \beta y + \gamma z) = \bar{\beta} (x, y) + \bar{\gamma} (x, z), x, y, z \in X.$
- (viii) $(x, 0) = 0, \forall x \in X$

Define with respect to Hilbert space :

- (ix) Normal operator
- (x) Projection

Section-B

4 each

2. If C is a base for the topological space X and Y is a subspace of X , then show that $A = \{B \cap Y : B \in C\}$ is a base for Y .

Or

Let X and Y be topological spaces and $f : X \rightarrow Y$. Then show that f is continuous iff $f^{-1}(B^0) = [f^{-1}(B)]^0$ for every subset B of Y .

3. Show that the property of a space being a T_0 -space is hereditary.

Or

Show that the property of being a T_1 -space is a topological property.

4. Show that the subset M of a normed linear space N is bounded if and only if there is a positive constant K such that :

$$\|x\| \leq K \forall x \in M$$

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(2)

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Or

Let $T : N \rightarrow N'$ be a linear transformation. Then show that T is a bounded if and only if T maps bounded sets in N into bounded sets in N' , where N and N' are normed linear spaces over the same field.

5. If S is a non-empty subset of a Hilbert space H , then show that S^\perp is a closed linear subspace of H . Is S^\perp a Hilbert space? Why? 3+1=4

Or

Let $\{e_1, e_2, \dots, e_n\}$ be a finite orthonormal set in a Hilbert space H . If x is any vector in H , then show that :

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2$$

6. In a Hilbert space H , show that :

$$\|T^* T\| = \|T\|^2$$

where T^* is adjoint of operator T .

Or

If $\{T_n\}$ is a sequence of self adjoint operators on a Hilbert space H and if $\{T_n\}$ converges to an operator T , then show that T is self adjoint.

Section-C

20 each

7. Let X and Y be topological spaces and $f: X \rightarrow Y$ be a bijective map. Show that the following are equivalent :
- (i) f is open and continuous
 - (ii) f is homeomorphism
 - (iii) f is closed and continuous

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8. Show that in a metric space (X, d) the following are equivalent :
- (i) X is compact
 - (ii) X is limit point compact
 - (iii) X is sequentially compact
9. Let M be a linear subspace of normed linear space N and f be a functional defined on M . If x_0 is a vector not in M and if M_0 is the subspace spanned by M and x_0 , then show that f can be extended to a functional f_0 defined on M_0 such that :

$$\|f_0\| = \|f\|$$

10. Let H be a Hilbert space and f be an arbitrary functional in H^* . Then show that there exists a unique vector y in H such that :

$$f(x) = (x, y) \quad \forall x \in H \text{ and } \|f\| = \|y\|$$

11. Let H be a given Hilbert space and T^* be adjoint of the operator T on H . Then show that T^* is a bounded linear transformation and T determines T^* uniquely.